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On Liouville integrability of zero-curvature equations and the Yang hierarchy†

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Abstract. Sufficient conditions for a zero-curvature equation $U_t - V_x + [U, V] = 0$ being Liouville integrable are investigated. In the case that the equation is integrable an explicit formula of the Poisson bracket $\{H(\lambda), H(\mu)\}$ for Hamiltonians H is proposed. The Yang hierarchy is derived and shown to be Liouville integrable.

1. Introduction

An elegant geometrical theory of finite-dimensional Hamiltonian systems has been built since the late 1960s [1, 2]. The beautiful Liouville-Arnold theorem represents a landmark of the theory. According to this theorem a Hamiltonian system with n degrees of freedom is completely integrable if it possesses n integrals which are involution in pairs. Much progress has been made in the last twenty years in extending this theorem to infinite-dimensional systems. However, there is still a long way to go. Accordingly an authoritative definition of complete integrability of an infinite-dimensional system is not available in current literature. In this paper we adopt two working definitions which are nowadays quite popular. First we call a nonlinear evolution equation (NLEE) Lax integrable if it admits a zero-curvature representation:

$$U_t - V_x + [U, V] = 0 \tag{1.1}$$

where U, V are matrices belonging to a Lie algebra. This definition is extensively adopted in soliton theory (see, e.g., [3]). Secondly we call an NLEE Liouville integrable [4] if it can be written as a generalised Hamiltonian equation $u_t = J\,\delta H/\delta u$ with a well defined Poisson bracket $\{\,,\}$, and it possesses an infinite number of conserved densities $\{H_n\}$ which are involution in pairs $\{H_n, H_m\} = 0$. Both of the above two definitions are formal. However, most of the known systems, which are integrable in either of the above-mentioned senses, do present peculiar features which make them much more like finite-dimensional integrable systems. Two problems remain open.

(i) A full description of those matrices U for which there exists a hierarchy of non-trivial zero-curvature equations

$$U_{t_n} - V^{(n)} + [U, V^{(n)}] = 0. \tag{1.2}$$

(ii) A formulation of sufficient and/or necessary conditions under which the above zero-curvature equations are Liouville integrable.

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The first problem amounts, in fact, to one of generating Lax integrable systems. This is a central, difficult, subject in soliton theory. The aim of the present paper is to give a partial answer to the two above important problems. First, we propose a model class of matrices $U = U(\lambda)$ (2.17). Second, we give a sufficient condition under which there exists matrices $V^{(n)}$ such that the hierarchy of equations (1.2) represent non-trivial evolution equations. Third, we formulate sufficient conditions for this hierarchy to be Liouville integrable. The key result is an explicit formula for the Poisson bracket (3.17). We also propose a simple method for proving the locality of the hierarchy of equations (1.2).

This paper is divided into five sections. The next section contains a brief description of basic notions. After doing that we explain the motivation of the paper, and establish the main results in § 3. Two illustrative examples, the Giachetti–Johnson (GJ) and the Jaulent-Miodek (JM) hierarchies, are given in § 4. Section 5 is devoted to the derivation of the Yang hierarchy and the proof of its Liouville integrability.

2. Preliminaries

2.1. Basic notation

Let G be a matrix Lie algebra over the complex field $\mathbb{C}$, and $\tilde{G} = G \otimes \mathbb{C}(\lambda, \lambda^{-1})$ be its loop algebra, where $\mathbb{C}(\lambda, \lambda^{-1})$ is the set of Laurent polynomials in λ. The gradation of $\tilde{G}$ is taken by

$$\deg(x \otimes \lambda^n) = n \qquad x \in G. \tag{2.1}$$

Let $g \in \tilde{G}$ and $g = \Sigma_n g_n$, $\deg g_n = n$, be its gradation decomposition. We set

$$g_+ = \sum_{n \geq 0} g_n. \tag{2.2}$$

The scalar product of two vectors $F = (F_i)$ and $F' = (F'_i)$ is denoted by

$$F \cdot F' = \sum_i F_i F'_i. \tag{2.3}$$

In the same way the scalar product of two matrices $A = (A_{ij})$ and $B = (B_{ij})$ is defined by

$$A \cdot B = \sum_{ij} A_{ij} B_{ij} = \langle A^{\mathrm{T}}, B \rangle \tag{2.4}$$

where T represents the transpose and

$$\langle A, B \rangle \equiv \mathrm{Tr}(AB) = \sum_{ij} A_{ij} B_{ji}. \tag{2.5}$$

Let S be the Schwartz space over $\mathbb{R} = (-\infty, \infty)$, $S^p \equiv S \otimes \cdots \otimes S$ (p times). The operator $\partial = \mathrm{d}/\mathrm{d}x$ introduces an equivalence relation among elements of S^p:

$$f = g (\mathrm{mod}\, \partial) \Leftrightarrow \exists h \text{ such that } f - g = \partial h \qquad f, g, h \in S^p. \tag{2.6}$$

The equivalence class which contains f is denoted by [5]

$$\int f \, \mathrm{d}x = \{f + \partial h \,|\, h \in S^p\}. \tag{2.7}$$

In the following discussion the algebra $\tilde{G}$ will be extended to

$$G(S) = \left\{ \sum f_i e_i \,\middle|\, e_i \in \tilde{G}, f_i \in S \right\}. \tag{2.8}$$

The equivalence relation $A = B(\text{mod}\,\partial)$ and the equivalence class $\int A\,\mathrm{d}x$, for $A, B \in G(S)$, are defined in a similar way.

2.2. Generalised Hamiltonian equations

Let

$$u = u(x, t) = (u_1, \ldots, u_p)^{\mathrm{T}} \tag{2.9}$$

be a smooth vector function which belongs to S^p for any fixed t. A linear operator $J = J(u)$: $S^p \to S^p$ is called symplectic (or implectic [6], Hamiltonian [5, 7]) if (i) it is skew-symmetric with respect to the inner product

$$(F, G) = \int F \cdot G\,\mathrm{d}x \qquad F, G \in S^p \tag{2.10}$$

i.e.

$$(JF, G) = -(F, JG); \tag{2.11}$$

and (ii) it holds that

$$(J'(u)[Jf]g, h) + (J'(u)[Jg]h, f) + (J'(u)[Jh]f, g) = 0 \tag{2.12}$$

for any $f, g, h \in S^p$, where

$$J'(u)[f] = (\mathrm{d}/\mathrm{d}\varepsilon)J(u + \varepsilon f)\big|_{\varepsilon=0}.$$

It is shown that [6], if J is symplectic, then the bracket

$$\{H, I\} = (\delta H/\delta u, J\,\delta I/\delta u) = \int ((\delta H/\delta u) \cdot (J\,\delta I/\delta u))\,\mathrm{d}x \tag{2.13}$$

is a well defined Poisson bracket between scalar functions $H, I \in S$. In this case the equation

$$u_t = J\,\delta H/\delta u \tag{2.14}$$

is called a (generalised) Hamiltonian equation with the Hamiltonian H, where

$$\delta/\delta u = (\delta/\delta u_1, \ldots, \delta/\delta u_p)^{\mathrm{T}} \tag{2.15a}$$

$$\delta/\delta u_i = \sum_{n \geq 0} (-\partial)^n (\partial/\partial u_i^{(n)}) \qquad u_i^{(n)} = \partial^n u_i \tag{2.15b}$$

stands for variational derivatives.

2.3. An isospectral problem

Consider the isospectral problem

$$\psi_x = U(u, \lambda)\psi \tag{2.16}$$

with

$$U = U(u, \lambda) = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_p e_p(\lambda) \tag{2.17}$$

where $u = (u_1, \ldots, u_p) \in S^p$ and $e_0(\lambda), \ldots, e_p(\lambda) \in \tilde{G}$. We suppose that $e_0, e_1, \ldots, e_p$ are linearly independent, and

$$\varepsilon_0 > 0 \qquad \varepsilon_0 > \varepsilon_i \qquad i = 1, \ldots, p \tag{2.18}$$

where $\varepsilon_i = \deg e_i$, $i = 0, 1, \ldots, p$. With the above assumptions we can define the rank for ∂, u_i, λ and $A \in \tilde{G}$ in such a way that [8] (i) if ab makes sense for two entities a and b, then $\mathrm{rank}(ab) = \mathrm{rank}(a) + \mathrm{rank}(b)$; (ii) the matrix U is of homogeneous rank, i.e. $\mathrm{rank}(e_0) = \mathrm{rank}(u_1 e_1) = \ldots = \mathrm{rank}(u_p e_p)$. To this end we take

$$\mathrm{rank}(A) = \deg(A) \qquad A \in \tilde{G}, \tag{2.19a}$$

$$\mathrm{rank}(\lambda) = \deg(A\lambda) - \deg(A) \tag{2.19b}$$

$$\mathrm{rank}(u_i) = \varepsilon_0 - \varepsilon_i \qquad i = 1, \ldots, p \tag{2.19c}$$

$$\mathrm{rank}(\partial) = \varepsilon_0 \tag{2.19d}$$

$$\mathrm{rank}(\beta) = 0 \qquad \beta = \text{constant}, \ \beta \neq 0. \tag{2.19e}$$

We observe that (2.19b) is well defined, because we have $\deg(A\lambda) + \deg(B) = \deg(A) + \deg(\lambda B) = \deg(\lambda[A, B])$. Hence $\deg(A\lambda) - \deg(A) = \deg(B\lambda) - \deg(B)$, and the right-hand side of (2.19b) is indeed independent of $A \in \tilde{G}$.

2.4. *A scheme for generating Lax integrable systems*

We have proposed in [9] a scheme for generating Lax integrable systems. Let an isospectral problem be given by (2.16) and (2.17). First, we take a solution $V = V(\lambda)$ of the equation

$$V_x(\lambda) = [U(\lambda), V(\lambda)]. \tag{2.20}$$

Second, we search for $\Delta_n \in \tilde{G}$ such that, for

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n \tag{2.21}$$

it holds that

$$V_x^{(n)} - [U, V^{(n)}] = \mathbb{C} e_1 + \ldots + \mathbb{C} e_p. \tag{2.22}$$

This requirement yields a hierarchy of evolution equations

$$U_{t_n}(\lambda) = V_x^{(n)}(\lambda) - [U(\lambda), V^{(n)}(\lambda)] \tag{2.23}$$

where we use the time variables t_n with subindices, as suggested in [10], to emphasise the special choice $V = V^{(n)}$.

We shall always search for solutions $V(\lambda)$ of (2.20) which are of the form

$$V(\lambda) = \sum_{n \geqslant 0} V_n(u) \lambda^{-n} \tag{2.24}$$

with $V_0 = \text{constant} \neq 0$. Then by definition (2.2),

$$(\lambda^n V(\lambda))_+ = \sum_{n \geqslant i \geqslant 0} V_i \lambda^{n-i}. \tag{2.25}$$

To write the equations more concisely, we adopt in subsequent sections, unless otherwise specified, the following notations. For matrices $F(\lambda) \in G(S)$ we write

$$F_t(\lambda; \mu) = \sum_{n \geqslant 0} F_{t_n}(\lambda) \mu^{-n}. \tag{2.26}$$

In the same way we write

$$(u_i(\mu))_t = \sum_{n \geqslant 0} (u_i)_{t_n} \mu^{-n}. \tag{2.27}$$

2.5. The trace identity

We proved in [11, 12] the following result. Suppose that for any integer m the solution V of (2.20), which is of rank m, is unique up to a constant multiplier. Then for any solution V of homogeneous rank, there exists a constant γ such that

$$(\delta/\delta u_i)\langle V, \partial U/\partial\lambda\rangle = (\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})\langle V, \partial U/\partial u_i\rangle. \tag{2.28}$$

We shall suppose in subsequent sections that the above-mentioned condition which leads to (2.28) is fulfilled. In this case we set

$$I = \langle V, \partial U/\partial\lambda\rangle \tag{2.29}$$

and define a scalar $H = H(\lambda)$ by the equation

$$(\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})H = I. \tag{2.30}$$

The trace identity (2.28) then yields that

$$\delta H/\delta u_i = \langle V, \partial U/\partial u_i\rangle. \tag{2.31}$$

Let $\{H_n\}$ be defined by

$$H(\lambda) = \sum_{n\geqslant 0} H_n\lambda^{-n}. \tag{2.32}$$

We shall show in the next section that the set $\{H_n\}$ provides, under certain conditions, conserved densities of the hierarchy of equations connected with the isospectral problem (2.16) and (2.17).

3. Liouville integrability of zero-curvature equations

First we explain the motivation of the present research. It is known that for most isospectral problems (2.16) and (2.17) the corresponding hierarchy (1.2) consists of bi-Hamiltonian systems, i.e. each equation in the hierarchy can be written as a generalised Hamiltonian equation in two different ways:

$$u_{t_n} = J\,\delta H_n/\delta u = J_1\,\delta H_{n-1}/\delta u$$

where J, J_1 and their linear combination $\alpha J + \beta J_1$ are symplectic operators for constants α and β. In this case, if J is invertible then, by setting $L = J_1 J^{-1}$, we have

$$u_{t_n} = J\,\delta H_n/\delta u = JL\,\delta H_{n-1}/\delta u$$

and thus

$$u_{t_n} = JL^n f(u)$$

for some function f. The operator L is called a recursion operator for the hierarchy (1.2), and its conjugate L^* is called a hereditary symmetry [13] that plays an important role in generating symmetries of the hierarchy of equations. There was even a long-standing conjecture that all recursion operators derived from isospectral problems are conjugates of hereditary symmetries. By making use of the trace identity (2.28) and the skew-symmetry of J and $J_1 = JL$, i.e.

$$J^* = -J \qquad JL = L^*J$$

we had successfully established in [8] the Liouville integrability of various hierarchies of equations (1.2). To our surprise, when we try to apply the same method to the Giachetti-Johnson (GJ) hierarchy (see § 4) and the Yang hierarchy (see § 5) great difficulty arose: we did find J and L such that $J^* = -J$ and $JL = L^*J$, but J is not symplectic; we can also find J and L such that J is symplectic and the whole hierarchy can be written as $u_t = J\,\delta H_n/\delta u = JL^n f(u)$. However $JL \neq L^*J$. This dilemma forces us to think that perhaps GJ and Yang hierarchies might be counterexamples to the conjecture. We then decided to search for an alternative approach. Finally we succeeded in establishing an explicit formula for the Poisson bracket $\{H(\lambda), H(\mu)\} = (\mathrm{d}/\mathrm{d}x)P(\lambda, \mu)$ (see (3.17)) that implies that $\{H_n, H_m\} = 0 (\mathrm{mod}\ \partial)$ for $n, m \geqslant 0$, as needed by the Liouville integrability. We then realised that this formula also applies to the bi-Hamiltonian systems as well. Therefore the formula for Poisson brackets, together with the trace identity, provide us effective tools for establishing the Liouville integrability of general zero-curvature equations. In a subsequent paper we successfully proved that the stationary zero-curvature equation (2.20) can be cast in completely integrable Hamiltonian system. This result paves a way for constructing a large class of finite-dimensional integrable Hamiltonian systems. We begin with the following simple propositions.

Proposition 1. Let $V = V(\lambda)$ be a solution of (2.20) and (2.24), and $(\lambda^n V)_+$ be defined by (2.25). Then it holds that

$$\sum_{n\geqslant 0} (\lambda^n V(\lambda))_+ \mu^{-n} = \mu V(\mu)/(\mu - \lambda) \tag{3.1}$$

and

$$\sum_{n\geqslant 0} ((\lambda^n V(\lambda))_{+x} - [U(\lambda), (\lambda^n V(\lambda))_+])\mu^{-n} = [\mu(U(\mu) - U(\lambda))/(\mu - \lambda), V(\mu)]. \tag{3.2}$$

Proof. We have

$$\begin{aligned}
\sum_{n\geqslant 0} (\lambda^n V(\lambda))_+ \mu^{-n}
&= \sum_{n\geqslant 0} \left(\sum_{n\geqslant i\geqslant 0} \lambda^{n-i} V_i \right) \mu^{-n} \\
&= \sum_{i\geqslant 0} \left(\sum_{n\geqslant i} (\lambda/\mu)^n \right) (V_i \lambda^{-i}) \\
&= (1 - \lambda/\mu)^{-1} \sum_{i\geqslant 0} [(\lambda/\mu)^i (V_i \lambda^{-i})] \\
&= (1 - \lambda/\mu)^{-1} \sum_{i\geqslant 0} V_i \mu^{-i} \\
&= [\mu/(\mu - \lambda)] V(\mu)
\end{aligned}$$

which proves (3.1). Making use of (3.1) we deduce further that

$$\begin{aligned}
\sum_{n\geqslant 0} ((\lambda^n V(\lambda))_{+x} - [U(\lambda), (\lambda^n V(\lambda))_+])\mu^{-n}
&= [\mu/(\mu - \lambda)](V_x(\mu) - [U(\lambda), V(\mu)]) \\
&= [\mu/(\mu - \lambda)]([U(\mu), V(\mu)] - [U(\lambda), V(\mu)]) \\
&= [\mu(U(\mu) - U(\lambda))/(\mu - \lambda), V(\mu)].
\end{aligned}$$

The proof is completed.

By using the above proposition and the concise notation (2.26), the hierarchy of equations (2.23) can be put together as follows.

Proposition 2. The hierarchy (2.23) with (2.21) can be written as

$$U_t(\lambda;\mu)=[\mu(U(\mu)-U(\lambda))/(\mu-\lambda),\ V(\mu)]+\Delta_x(\mu)-[U(\lambda),\Delta(\mu)] \tag{3.3}$$

where the concise notation (2.26) has been used and

$$\Delta(\mu)\equiv\sum_{n\geq 0}\Delta_n\mu^{-n}. \tag{3.4}$$

Proof. By (2.26), (2.23) and (2.21) we have

$$\begin{aligned}U_t(\lambda;\mu)&=\sum_{n\geq 0}U_{t_n}(\lambda)\mu^{-n}\\&=\sum_{n\geq 0}(V_x^{(n)}(\lambda)-[U(\lambda),V^{(n)}(\lambda)])\mu^{-n}\\&=\sum_{n\geq 0}((\lambda^nV(\lambda))_{+x}-[U(\lambda),(\lambda^nV(\lambda))_+])\mu^{-n}+\sum_{n\geq 0}(\Delta_{nx}-[U(\lambda),\Delta_n])\mu^{-n}\end{aligned}$$

which, together with (3.2), imply the desired conclusion.

The following theorem is an immediate consequence of (3.3).

Theorem 3. Let the matrix $U(\lambda)$ be defined by (2.17). If there exists a matrix $\Delta(\mu)$ and p independent functions $f_1(\mu,u),\ldots,f_p(\mu,u)$ such that

$$[\mu(U(\mu)-U(\lambda))/(\mu-\lambda),\ V(\mu)]+\Delta_x(\mu)-[U(\lambda),\Delta(\mu)]=\sum_{p\geq i\geq 1}f_i(\mu,u)e_i(\lambda). \tag{3.5}$$

Then one can relate the isospectral problem (2.16) with the following hierarchy of NLEE:

$$u_{t_n}=(f_{1n},\ldots,f_{pn})^{\mathrm{T}} \tag{3.6}$$

where u is given by (2.9) and f_{in} are defined by

$$f_i(\mu,u)=\sum_{n\geq 0}f_{in}(u)\mu^{-n}. \tag{3.7}$$

We observe that by (2.27) the hierarchy (3.6) can be written as

$$(u(\mu))_t=f(u) \tag{3.8}$$

where $f(u)=(f_1,\ldots,f_p)^{\mathrm{T}}$.

The following proposition is essential.

Proposition 4. Let the matrix U be defined by (2.17). Suppose that there exist matrices V and $\bar{V}$ which satisfy, respectively, (2.20) and

$$U_\tau=\bar{V}_x-[U,\bar{V}]. \tag{3.9}$$

Then it holds that

$$\sum_i\langle V,\partial U/\partial u_i\rangle u_{i\tau}=\langle V,\bar{V}\rangle_x. \tag{3.10}$$

Proof. Let $b_1, \ldots, b_N$ be a base of G and the structure constants C_{ij}^k be defined by $[b_i, b_j] = \Sigma_k C_{ij}^k b_k$. Suppose that $V = \Sigma\, v_i b_i$, $\bar{V} = \Sigma\, \bar{v}_i b_i$ and $e_i = \Sigma_{N \geq j \geq 1} e_{ij}(\lambda) b_j$, where $e_{ij}(\lambda) \in \mathbb{C}(\lambda, \lambda^{-1})$. Taking $u_0 = 1$ for convenience we then have

$$U = \sum_{N \geq j \geq 1} \bar{u}_j b_j \qquad \bar{u}_j = \sum_{p \geq i \geq 0} u_i e_{ij}$$

and

$$[U, V] = \left(\sum_i \bar{u}_i b_i, \sum_j v_j b_j\right) = \sum_{ijk} C_{ij}^k \bar{u}_i v_j b_k$$

which, along with the suppositions (2.20) and (3.9), give us

$$v_{kx} = \sum_{ij} C_{ij}^k \bar{u}_i v_j \qquad \bar{u}_{k\tau} = \bar{v}_{kx} - \sum_{ij} C_{ij}^k \bar{u}_i \bar{v}_j.$$

Let K_{ij} be defined by $K_{ij} = \langle b_i, b_j \rangle$. We observe that $K_{ij} = K_{ji}$ and

$$\sum_l K_{il} C_{jk}^l = \langle b_i, [b_j, b_k] \rangle = \langle b_j, [b_k, b_i] \rangle$$

$$= \sum_l K_{jl} C_{ki}^l = -\sum_l K_{jl} C_{ik}^l.$$

Hence

$$\begin{aligned}
\sum_i \langle V, \partial U/\partial u_i \rangle u_{i\tau} &= \sum_i \left\langle \sum_l v_l b_l, \sum_k e_{ik} b_k \right\rangle u_{i\tau} \\
&= \sum_{lk} K_{lk} V_l \left(\sum_i e_{ik} u_{i\tau}\right) = \sum_{lk} K_{lk} v_l \bar{u}_{k\tau} \\
&= \sum_{lk} K_{lk} v_l \Big(\bar{v}_{kx} - \sum_{ij} C_{ij}^k \bar{u}_i \bar{v}_j\Big) \\
&= \sum_{lk} K_{lk} v_l \bar{v}_{kx} - \sum_{ijlk} K_{jl} C_{ik}^l v_j \bar{u}_i \bar{v}_k \\
&= \sum_{lk} K_{lk} v_l \bar{v}_{kx} + \sum_{ijlk} K_{lk} C_{ij}^l \bar{u}_i v_j \bar{v}_k \\
&= \sum_{l,k} K_{lk} (v_l \bar{v}_{kx} + v_{lx} \bar{v}_k) = \left(\sum_{lk} K_{lk} v_l \bar{v}_k\right)_x \\
&= \left\langle \sum v_l b_l, \sum \bar{v}_k b_k \right\rangle_x = \langle V, \bar{V} \rangle_x.
\end{aligned}$$

The proof is thus completed.

Proposition 5. Let the matrix U, the corresponding hierarchy and the scalar function $H(\lambda)$ be defined, respectively, by (2.17), (2.23) and (2.30). Then it holds that

$$\sum_i (\delta H/\delta u_i)(u_i(\mu))_t = \langle V(\lambda), \mu V(\mu)/(\mu - \lambda) + \Delta(\mu) \rangle \tag{3.11}$$

and

$$\mu \langle V(\lambda), V(\mu) \rangle_x / (\mu - \lambda) = \mu \langle (U(\mu) - U(\lambda))/(\mu - \lambda), [V(\mu), V(\lambda)] \rangle. \tag{3.12}$$

Proof. By (2.31), (2.27), proposition 4, (3.1) and (3.4), we see that

$$\begin{aligned}\sum_i (\delta H/\delta u_i)(u_i(\mu))_t &= \sum_{ni} \langle V, \partial U/\partial u_i\rangle (u_i)_{t_n}\mu^{-n} \\ &= \sum_n \langle V, V^{(n)}\rangle_x \mu^{-n} \\ &= \sum_n \langle V, (\lambda^n V)_+ + \Delta_n\rangle \mu^{-n} \\ &= \langle V(\lambda), \mu V(\mu)/(\mu-\lambda) + \Delta(\mu)\rangle_x\end{aligned}$$

which proves (3.11). To prove (3.12) we write for convenience

$$U = U(\lambda) \qquad V = V(\lambda) \qquad U' = U(\mu) \qquad V' = V(\mu). \tag{3.13}$$

Then we have

$$\begin{aligned}\langle V, V'\rangle_x &= \langle V_x, V'\rangle + \langle V, V'_x\rangle \\ &= \langle [U, V], V'\rangle + \langle V, [U', V']\rangle = \langle U - U', [V, V']\rangle\end{aligned}$$

which implies (3.12) and the proof is completed.

Here, and in the following, to simplify the notation we shall always write $f' = f(\mu)$ if $f = f(\lambda)$. Thus, for example,

$$\begin{array}{llll} a = a(\lambda) & b = b(\lambda) & c = c(\lambda) & H = H(\lambda) \\ a' = a(\mu) & b' = b(\mu) & c' = c(\mu) & H' = H(\mu) \end{array} \tag{3.14}$$

and so on.

Suppose now that there exists an operator $J: S^p \to S^p$ such that

$$J\lambda^k(\langle V, \partial U/\partial u_1\rangle, \ldots, \langle V, \partial U/\partial u_p\rangle)^{\mathrm{T}} = (f_1(\lambda, u), \ldots, f_p(\lambda, u))^{\mathrm{T}} \tag{3.15}$$

where f_i are defined by (3.6) or (3.8). Then the left-hand side of (3.11) is

$$\begin{aligned}\sum_i (\delta H/\delta u_i)(u_i(\mu))_t &= (\delta H/\delta u)(f(\mu, u)) \\ &= \mu^k (\delta H/\delta u)(J[\langle V', \partial U'/\partial u_i\rangle]) \\ &= \mu^k (\delta H/\delta u)(J\,\delta H'/\delta u).\end{aligned} \tag{3.16}$$

Thus we deduce the following.

Proposition 6. Under the supposition (3.15) we have

$$\mu^k(\delta H(\lambda)/\delta u)(J\,\delta H(\mu)/\delta u) = \langle V(\lambda), \mu V(\mu)/(\mu-\lambda) + \Delta(\mu)\rangle_x \tag{3.17}$$

and consequently

$$\{H(\lambda), H(\mu)\} = \int (\delta H(\lambda)/\delta u)(J\,\delta H(\mu)/\delta u)\,\mathrm{d}x = 0. \tag{3.18}$$

That is, the set $\{H_n\}$ are involution in pairs with respect to $\{\ ,\ \}$,

$$\{H_m, H_n\} = 0 \qquad \forall m, n \geqslant 0 \tag{3.19}$$

where $\{H_n\}$ are defined by (2.32).

We observe that by (3.16) the hierarchy (3.8) takes the form

$$(u(\mu))_t = J\delta(\mu^k H(\mu))/\delta u \tag{3.20}$$

or equivalently

$$u_{t_n} = J\delta H_{n+k}(u)/\delta u. \tag{3.21}$$

Furthermore by (3.19) we have

$$(H_m(u))_{t_n} = \{H_m, H_n\} = 0.$$

Hence $\{H_m\}$ are common conserved densities of the hierarchy of equations (3.21).

The following theorem gives a summation of the above propositions.

Theorem 7. Let the isospectral problem be given by (2.16) and (2.17) and let V be a solution of (2.20). Suppose that (i) equation (3.5) holds for a matrix $\Delta(\mu)$ and p independent functions $f_1(\mu, u), \ldots, f_p(\mu, u)$; (ii) the operator J defined by (3.15) is symplectic. Then we have the following conclusions.

(*a*) The scalar function $H(\lambda, u)$ defined by (2.30) satisfies (2.31).

(*b*) The evolution equations (3.6) can be written in Hamiltonian form (3.21).

(*c*) The formula (3.17) for Poisson bracket holds.

(*d*) The hierarchy of equations (3.21) are Liouville integrable and the set $\{H_m\}$ constitutes the common set of infinitely many conserved densities which are involution in pairs.

It is known that for most isospectral problems (2.16) and (2.17), the hierarchy (3.21) can be written in the form

$$u_{t_n} = JL^n g(u) \tag{3.22}$$

where $g \in S$ and L is an operator which is usually an integro-differential operator. Moreover, the conjugate of L, denoted by L^*, is a hereditary symmetry [13].

There is a peculiar and intriguing thing, which puzzled me for some years, that equations (3.22) are usually local, i.e. they are pure differential equations in spite of the fact that L is actually non-local. To prove this fact different authors used different techniques (see, e.g. [14–16]). However no unified and satisfactory explanation is proposed in the literature so far as we know. We find that the following elementary proposition is very useful to establish the above-mentioned local property.

Proposition 8. Let $V = V(\lambda, u)$ be a solution of (2.20). Then we have

$$(\mathrm{d}/\mathrm{d}x)(\det V(\lambda, u)) = 0. \tag{3.23}$$

Proof. Let $V = (V_{ij})$, $U = (U_{ij})$ be $N \times N$ matrices and S_N be the set of all permutations of $(1, 2, \ldots, N)$. By supposition we have

$$V_{ijx} = \sum_k (U_{ik}V_{kj} - V_{ik}U_{kj}).$$

Hence

$$(\det V)_x = \sum_{\sigma \in S_N} (-1)^\sigma (V_{1\sigma(1)}V_{2\sigma(2)} \ldots V_{N\sigma(N)})_x$$
$$= \sum_{ik\sigma} (-1)^\sigma (V_{1\sigma(1)} \ldots V_{i-1,\sigma(i-1)})(U_{ik}V_{k\sigma(i)} - V_{ik}U_{k\sigma(i)}) \ldots$$
$$(V_{i+1,\sigma(i+1)} \ldots V_{N\sigma(N)}) = (\det V)\sum_{i\sigma}(U_{ii} - U_{\sigma(i)\sigma(i)}) = 0.$$

The special case that $V \in \mathrm{sl}(2)$ is of particular interest. In this case we have the next proposition.

Proposition 9. Let

$$V(\lambda, u) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{n \geq 0} V_n(u)\lambda^{-n}$$

be a solution of (2.20). Suppose that (i) $a_0 = \alpha = \text{constant} \neq 0$, and b_0 and c_0 are local; (ii) b_{n+1} and c_{n+1} can be obtained from b_i, c_i and a_i for $i \leq n$ by algebraic and differential manipulations. Then all a_i, b_i and c_i are local.

Proof. By proposition 9 we have

$$a^2 + bc = \beta = \text{constant}. \tag{3.24}$$

Therefore

$$a_n = \left(\beta - \sum_{n-1 \geq i \geq 1} (a_i a_{n-i}) - \sum_{n \geq i \geq 0} (b_i c_{n-i})\right) \alpha^{-1}$$

which shows if b_i, c_i ($i \leq n$) and a_i ($i \leq n-1$) are local then a_n is also local. Now by (i) a_0, b_0 and c_0 are local and by (ii), if b_i, c_i, a_i are local for $i \leq n$ then b_{n+1} and c_{n+1} are local. An obvious induction on n thus completes the proof.

4. Two examples

4.1. Giachetti–Johnson (GJ) hierarchy

Let $G = \mathrm{sl}(2)$ with the base

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.1}$$

Consider the isospectral problem (2.16) [17] with

$$U = -\lambda h + qe + rf + sh \tag{4.2}$$

where $u_1 = q(x, t)$, $u_2 = r(x, t)$ and $u_3 = s(x, t)$ represent three potentials. Let

$$V = ah + be + cf \tag{4.3}$$

be a solution of (2.20).

To derive the corresponding hierarchy of equations we take

$$\Delta(\mu) = \delta(\mu)h.$$

Then

$$\begin{aligned}[\mu(U(\lambda) - U(\mu))/(\lambda - \mu),\ V(\mu)] &= [-\mu h, b'e + c'f] \\ &= -2\mu b'e + 2\mu c'f\end{aligned}$$

and

$$\begin{aligned}\Delta_x(\mu) - [U(\lambda), \Delta(\mu)] &= \delta'_x h - [qe + rf, \delta' h] \\ &= \delta'_x h + 2q\,\delta' e - 2r\,\delta' f.\end{aligned}$$

Hence we have

$$f_1(\mu, u) = 2q\delta' - 2\mu b' \qquad f_2(\mu, u) = -2r\delta' + 2\mu c' \qquad f_3(\mu, u) = \delta'_x$$

and by theorem 7 the desired hierarchy is

$$q_{t_n} = -2b_{n+1} + 2q\delta_n \qquad r_{t_n} = 2c_{n+1} - 2r\delta_n \qquad s_{t_n} = \delta_{nx}. \tag{4.4}$$

Taking in particular $\delta' = \mu a'$ or $\delta_n = a_{n+1}$ the above hierarchy (4.4) reduces to

$$q_{t_n} = -2b_{n+1} + 2qa_{n+1} \qquad r_{t_n} = 2c_{n+1} - 2ra_{n+1} \qquad s_{t_n} = a_{n+1x}. \tag{4.5}$$

To write the above hierarchy of equations in Hamiltonian forms we observe that

$$\langle e, f\rangle = 1 \qquad \langle h, h\rangle = 2.$$

Hence

$$\langle V, \partial U/\partial q\rangle = \langle ah + be + cf, e\rangle = c$$

$$\langle V, \partial U/\partial r\rangle = b \qquad \langle V, \partial U/\partial s\rangle = 2a.$$

It is easy to verify that

$$J(\mu(c', b', 2a')^{\mathrm{T}}) = (-2\mu b' + 2\mu q a', 2\mu c' - 2\mu r a', \mu a'_x) \tag{4.6}$$

where

$$J = \begin{pmatrix} 0 & -2 & q \\ 2 & 0 & -r \\ -q & r & \partial \end{pmatrix} \tag{4.7}$$

which is, as is easily seen, a symplectic operator [17]. Therefore by theorem 7 the above hierarchy (4.5) takes the form

$$u_{t_n} = J\,\delta H_{n+1}/\delta u \tag{4.8}$$

with

$$\delta H/\delta u = (c_n, b_n, 2a_n) \tag{4.9}$$

and equations (4.8) are Liouville integrable. The generating function $H(\lambda, u)$ of H_n can be calculated from the equation

$$(\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})H(\lambda) = \langle V, \partial U/\partial\lambda\rangle.$$

Since $\langle V, \partial U/\partial\lambda\rangle = -2a$, we see that $H_{n+1} = 2a_{n+2}/(n+1-\gamma)$. To fix the constant γ we observe that from (4.5), (4.6) and (4.8) we have

$$J(\delta/\delta u)[2a_{n+2}/(n+1-\gamma)] = (-2b_{n+1} + 2qa_{n+1}, 2c_{n+1} - 2ra_{n+1}, a_{n+1x}).$$

Setting in this equation $n = 0$ we see that $\gamma = 0$. Therefore

$$H_{n+1} = 2a_{n+2}/(n+1). \tag{4.10}$$

We note furthermore that from (2.20) one deduces that

$$a_0 = \alpha = \text{constant}$$

$$2b_{n+1} = -b_{nx} + 2sb_n + 2qa_n \qquad 2c_{n+1} = c_{nx} + 2sc_n - 2ra_n.$$

Hence by proposition 9 all equations (4.8) are pure differential equations.

The typical equation in the hierarchy (4.8) is [18]

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_\tau = \begin{pmatrix} 2q_x - 4sq + 2q^2 r \\ 2r_x + 4sr - 2qr^2 \\ (qr)_x \end{pmatrix} = J(\delta/\delta u)[(qr_x - rq_x)/2 + 4qrs].$$

Remark. As a verification of (3.17) we calculate that

$$\langle V(\lambda), V(\mu)\rangle = \langle ah+be+cf, a'h+b'e+c'f\rangle = 2aa'+bc'+b'c$$

$$\langle V(\lambda), \Delta(\mu)\rangle = \langle ah+be+cf, \mu a'h\rangle = 2\mu aa'.$$

Then (3.17) is, in the present case,

$$(c, b, 2a)^{\mathrm{T}} J(a', b', 2a')^{\mathrm{T}} = [(2aa'+bc'+cb')/(\mu-\lambda)+2aa']_x \tag{4.11}$$

which can be verified directly by using (4.7) and (2.20).

It is easy to verify from (4.7) that $(c, b, 2a)^{\mathrm{T}} J(a', b', 2a')^{\mathrm{T}} = 2(bc'-cb')+2(aa')_x$. Hence (4.11) reduces to

$$2(\lambda-\mu)(b'c-bc') = 2(aa')_x + (cb'+bc')_x \tag{4.12}$$

or by (4.9)

$$\begin{aligned}2(\lambda-\mu)[(\delta H(\lambda)/\delta q)(\delta H(\mu)/\delta r) - (\delta H(\lambda)/\delta r)(\delta H(\mu)/\delta q)]\\ = [(\tfrac{1}{2})(\delta H(\lambda/\delta s)(\delta H(\mu)/\delta s) + (\delta H(\lambda)/\delta q)(\delta H(\mu)/\delta r)\\ + (\delta H(\lambda)/\delta r)(\delta H(\mu)/\delta q)]_x.\end{aligned}$$

The same equation with $H(\lambda) = \ln(a(\lambda))$, where $a(\lambda)$ is the scattering coefficient of the isospectral problem, was obtained by Giachetti and Johnson [17].

4.2. The Jaulent–Miodek (JM) hierarchy

Consider the isospectral problem [19]

$$y_{xx} + (\lambda^2 - \lambda q - r)y = 0$$

that can be reduced to the model ones defined by (2.16) and (2.17) with

$$U = (q\lambda + r - \lambda^2)f + e.$$

To derive the corresponding hierarchy (1.2) we take the solution of (2.20) $V = ah+be+cf$. Then

$$\begin{aligned}a_x &= c + (\lambda^2 - q\lambda - r)b\\ b_x &= -2a\\ c_x &= 2(q\lambda + r - \lambda^2)a\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}[\mu(U(\lambda)-U(\mu))/(\lambda-\mu), V(\mu)] &= [\mu(q-\lambda-\mu)f, a'h+b'e+c'f]\\ &= \mu(q-\lambda-\mu)(2a'f-b'h)\\ &= \mu(\lambda+\mu-q)(b'_x f + b'h)\end{aligned}$$

where, as before, $a' = a(\mu)$, $b' = b(\mu)$, $c' = c(\mu)$. To cancel the term involving h we introduce

$$\Delta(\mu) = (b'\mu^2 - b'\mu q + \lambda\mu b')f$$

for which it holds that

$$\Delta_x(\mu) - [U(\lambda), \Delta(\mu)] = [(\mu^2 + \lambda\mu)b'_x - \mu(b'q)_x]f + \mu(q-\lambda-\mu)b'h.$$

Thus

$$[\mu(U(\lambda)-U(\mu))/(\lambda-\mu), V(\mu)]+\Delta(\mu)_x-[U(\lambda), \Delta(\mu)]$$
$$=(2\mu^2 b'_x-\mu[(b'q)_x+qb'_x]+2\lambda\mu b'_x)f$$

and we obtain

$$f_1(\lambda, u)=2\lambda b_x \qquad f_2(\lambda, u)=2\lambda^2 b_x-\lambda[(bq)_x+qb_x]. \tag{4.14}$$

Therefore the desired JM hierarchy is

$$q_t=2\mu b'_x \qquad r_t=2\mu^2 b'_x-\mu(2qb'_x+b'q_x)$$

or

$$q_{t_n}=2b_{n+1x} \qquad r_{t_n}=2b_{n+2x}-2qb_{n+1x}-q_x b_{n+1}. \tag{4.15}$$

To write (4.15) in its Hamiltonian form we note that

$$\langle V, \partial U/\partial\lambda\rangle=(q-2\lambda)b \qquad \langle V, \partial U/\partial q\rangle=b\lambda \qquad \langle V, \partial U/\partial r\rangle=b$$

the trace identity (2.28) then implies

$$(\delta/\delta q), \delta/\delta r)[(q-2\lambda)b]=(\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})(b\lambda, b)$$

or

$$(\delta/\delta q, \delta/\delta r)(qb_{n+1}-2b_{n+2})=(\gamma-n)(b_{n+1}, b_n). \tag{4.16}$$

This equation suggests we should search for the recursion relation among (b_{n+1}, b_n). Since from (4.13)

$$a_{nx}=c_n+b_{n+2}-qb_{n+1}-rb_n$$
$$b_{nx}=-2a_n$$
$$c_{nx}=2(qa_{n+1}+ra_n-a_{n+2}).$$

Then we can easily obtain $b_0=b_1=0$, $b_2=\beta=\text{constant}$ and

$$b_{n+2}=R(q)b_{n+1}+(R(r)-\partial^2/4)b_n$$

where

$$R(f)=f-\tfrac{1}{2}\partial^{-1}f_x.$$

From the above recursion relation we successively calculate

$$b_3=(\beta/2)q \qquad b_4=(\beta/8)(3q^2+4r)$$
$$b_5=(\beta/16)(5q^3+12qr-2q_{xx}).$$

Now setting $n=2$ in (4.16) we find that

$$(\gamma-1)(\beta, 0)=(\gamma-1)(b_2, b_1)=(\delta/\delta q, \delta/\delta r)(qb_2-2b_3)=0.$$

Thus $\gamma=1$ and we conclude that

$$(\delta/\delta u)H_n=(b_{n+1}, b_n)^{\mathrm{T}}$$

where $u=(q, r)^{\mathrm{T}}$, $H_1=\beta q$ and

$$H_n=(2b_{n+2}-qb_{n+1})/(n-1) \qquad n>1.$$

Therefore the JM hierarchy (4.15) takes the following Hamiltonian form:

$$u_{t_n} = J(b_{n+2}, b_{n+1})^{\mathrm{T}} = J\,\delta H_{n+1}/\delta u$$

where

$$J = \begin{bmatrix} 0 & 2\partial \\ 2\partial & -2\partial R(q) \end{bmatrix}.$$

We observe that

$$\lambda J(\langle V, \partial U/\partial q\rangle, \langle V, \partial U/\partial r\rangle) = \lambda J(b\lambda, b) = (2\lambda b_x, 2\lambda^2 b_x - 2\lambda q b_x - \lambda q_x b).$$

Comparing the above equation with (4.14) we see that (3.15) holds with $k=1$. Thus from (3.17) we deduce that

$$\begin{aligned}\{H(\lambda), H(\mu)\} &= (\langle V(\lambda), V(\mu)\rangle/(\mu-\lambda) + \langle V(\lambda), \Delta(\mu)/\mu\rangle)_x \\ &= \{[2a(\lambda)a(\mu) + b(\lambda)c(\mu) + b(\mu)c(\lambda)]/(\mu-\lambda) \\ &\quad + b(\lambda)b(\mu)(\lambda+\mu-q)\}_x\end{aligned}$$

which shows that each equation of the JM hierarchy is a Liouville integrable Hamiltonian equation.

5. Yang hierarchy

In a beautiful paper [20] Yang investigated the problem

$$(\omega\partial + Q)\psi = \lambda\psi \qquad \omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_1 \end{bmatrix} \tag{5.1}$$

where Q_1, Q_2, Q_3 are symmetric $N \times N$ matrices and I is the identity matrix of order N. The classical Sturm-Liouville theory on the spectral problem $(-\partial^2 + v)\psi = \lambda\psi$ was successfully extended by Yang to deal with the spectral problem (5.1).

Equation (5.1) can be written as (2.16) by taking

$$U = \omega Q - \lambda\omega = \begin{pmatrix} -Q_2 & -Q_3 + \lambda I \\ Q_1 - \lambda I & Q_2 \end{pmatrix}.$$

We shall discuss only the simple case $N=1$. In this case we set $Q_2 = -s$, $Q_3 = -r-q$, $Q_1 = r-q$. Then

$$U = \begin{pmatrix} s & (q+r)+\lambda \\ (r-q)-\lambda & -s \end{pmatrix}.$$

Taking now $G = \mathrm{sl}(2)$ with the base

$$e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have

$$[e_1, e_2] = -2e_3 \qquad [e_1, e_3] = -2e_2 \qquad [e_2, e_3] = -2e_1$$

$$\langle e_1, e_1\rangle = \langle e_3, e_3\rangle = -\langle e_2, e_2\rangle = 2$$

$$[a'e_3 + b'e_2 + c'e_1, ae_3 + be_2 + ce_1] = 2((a'b - b'a)e_1 + (a'c - c'a)e_2 + (b'c - c'b)e_3) \tag{5.2}$$

and

$$U=\lambda e_2+qe_2+re_1+se_3. \tag{5.3}$$

Let

$$V=ae_3+be_2+ce_1 \tag{5.4}$$

be a solution of (2.20). Then by (5.2) we obtain

$$\begin{aligned} a_x &= 2\lambda c+2(qc-rb) \\ b_x &= 2(sc-ra) \\ c_x &= -2\lambda a+2(sb-qa) \end{aligned} \tag{5.5}$$

or

$$\begin{aligned} 2c_{n+1} &= a_{nx}-2qc_n+2rb_n \\ 2a_{n+1} &= -c_{nx}-2qa_n+2sb_n \\ b_{nx} &= 2sc_n-2ra_n \end{aligned} \tag{5.6}$$

from which we calculate successively that

$$c_0=a_0=0 \qquad b_0=\beta=\text{constant} \tag{5.7a}$$

$$c_1=\beta r \qquad a_1=\beta s \qquad b_1=0 \tag{5.7b}$$

$$c_2=(\beta/2)(s_x-2qr) \qquad a_2=(\beta/2)(-r_x-2qs) \qquad b_2=(\beta/2)(s^2+r^2) \tag{5.7c}$$

$$\begin{aligned} c_3 &= (\beta/4)(-r_{xx}-4qs_x-2q_xs+4q^2r+2rs^2+2r^3) \\ a_3 &= (\beta/4)(-s_{xx}+4qr_x+2q_xr+4q^2r+2sr^2+2s^3) \\ b_3 &= (\beta/4)[2rs_x-2sr_x-4q(s^2+r^2)]. \end{aligned} \tag{5.7d}$$

By proposition 9 we assert that all a_i, b_i, c_i $(i\geqslant 0)$ are local.

To find the corresponding hierarchy we proceed as follows:

$$[\mu(U(\mu)-U(\lambda))/(\mu-\lambda),\ V(\mu)]=[\mu e_2,\ a'e_3+c'e_1]=-2\mu a'e_1+2\mu c'e_3. \tag{5.8}$$

Since the right-hand side of (5.8) does not contain a term fe_2 as needed to keep balance with U_t, we introduce

$$\Delta(\mu)=\delta(\mu)e_2$$

for which it holds that

$$\Delta'_x-[U,\Delta']=\delta'_xe_2+2r\,\delta'e_3-2s\,\delta'e_1.$$

By theorem 3 the hierarchy is then

$$q_t=\delta'_x \qquad r_t=-2\mu a'-2s\delta' \qquad s_t=2\mu c'+2r\delta'.$$

In a similar way as in the case of the GJ hierarchy we take $\delta'=\mu b'$. Then by (5.5)

$$\delta'_x=\mu b'_x=2\mu b'_x+2\mu ra'-2\mu sc'$$

and we obtain the desired Yang hierarchy:

$$q_t=2\mu b'_x+2\mu ra'-2\mu sc' \qquad r_t=-2\mu a'+2s\mu b' \qquad s_t=2\mu c'-2r\mu b'$$

or

$$q_{t_n}=2b_{n+1x}+2ra_{n+1}-2sc_{n+1} \qquad r_{t_n}=-2a_{n+1}+2sb_{n+1} \qquad s_{t_n}=2c_{n+1}-2rb_{n+1}. \tag{5.9}$$

To write (5.9) in their Hamiltonian form we note that

$$\langle V, \partial U/\partial\lambda\rangle = \langle ae_3 + be_2 + ce_1, e_2\rangle = -2b$$
$$\langle V, \partial U/\partial q\rangle = \langle V, e_2\rangle = -2b$$
$$\langle V, \partial U/\partial r\rangle = 2c \qquad \langle V, \partial U/\partial s\rangle = 2a.$$

By the trace identity we then obtain

$$(\delta/\delta q, \delta/\delta r, \delta/\delta s)(-2b) = (\lambda^{-\gamma}(\partial/\partial\lambda)\lambda^{\gamma})(-2b, 2c, 2a)$$

or

$$(\delta/\delta u)(-2b_{n+1}) = (\gamma - n)(-2b_n, 2c_n, 2a_n)^{\mathrm{T}} \tag{5.10}$$

where $u = (q, r, s)^{\mathrm{T}}$. To fix the constant γ we simply set $n = 0$ in (5.10) and find $0 = \gamma(-2\beta, 0, 0)$. Therefore $\gamma = 0$ and we conclude that

$$(\delta/\delta u)H_n = (-2b_n, 2c_n, 2a_n)^{\mathrm{T}} \qquad H_n = 2b_{n+1}/n. \tag{5.11}$$

According to the scheme suggested by theorem 7, it remains only to find a symplectic operator J such that

$$\lambda^k J(-2b, 2c, 2a) = (2\lambda b_x + 2\lambda ra - 2\lambda sc, 2s\lambda b - 2\lambda a, 2\lambda c - 2r\lambda b). \tag{5.12}$$

The operator J satisfying (5.12) with $k = 1$ is easily seen to be

$$J = \begin{pmatrix} -\partial & -s & r \\ s & 0 & -1 \\ -r & 1 & 0 \end{pmatrix}.$$

From (5.9), (5.11) and (5.12) we see that the Yang hierarchy (5.9) takes the form

$$u_{t_n} = J\,\delta H_{n+1}/\delta u \tag{5.13}$$

which is Liouville integrable by theorem 7, and since all a_i, b_i and c_i are local as shown above, the hierarchy (5.13) is pure differential.

To write explicitly formula (3.17) we have

$$\langle V(\lambda), V(\mu)\rangle = \langle ae_3 + be_2 + ce_1, a'e_3 + b'e_2 + c'e_1\rangle = 2(aa' - bb' + cc')$$
$$\langle V(\lambda), \Delta(\mu)\rangle = \langle ae_3 + be_2 + ce_1, \mu b'e_2\rangle = -2\mu bb'.$$

Thus

$$\{H(\lambda), H(\mu)\} = [2(aa' - bb' + cc')/(\mu - \lambda) - 2bb']_x.$$

As a final remark we mention that the explicit expression $P(\lambda, \mu)$, appearing in (3.17) for Poisson brackets $\{H(\lambda), H(\mu)\} = (P(\lambda, \mu))_x$, is not important in the present case of infinite-dimensional Hamiltonian equations. However, it plays an important role in the study of finite-dimensional integrable systems. In a subsequent paper [21] we have proved that, if the non-stationary equation $U_t - V_x + [U, V] = 0$ is Liouville integrable, then the stationary equations (2.20) $V_x = [U, V]$ are finite-dimensional Hamiltonian systems that are complete integrable in the strict Liouville sense. In the search for constants of motion of such systems, the explicit expression $P(\lambda, \mu)$ mentioned above then plays an essential role.

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References

[1] Abraham R and Marsden J E 1980 *Foundations of Mechanics* (New York: Benjamin/Cummings)
[2] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[3] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia: SIAM)
[4] Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490-2
[5] Gel'fand I M and Dikii L A 1976 *Funct. Anal Appl.* **10** no 4, 13-29
[6] Fuchssteiner B and Fokas A S 1981 *Physica* **4D** 47-66
[7] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[8] Tu G-z 1989 *J. Math. Phys.* **30** 330-8
[9] Tu G-z 1988 *Sci. Sin.* A **31** no 12, 28-39
[10] Flaschka H, Newell A C and Ratiu T 1983 *Physica* **9D** 300-32
[11] Tu G-z 1986 *Sci. Sin.* A **24** 138-48
[12] Tu G-z 1988 *Adv. Sci. China Ser. Math.* **2** 45-72
[13] Fuschssteiner B 1979 *Nonlinear Anal. TMA* **3** 849-62
[14] Wilson G 1981 *Ergod. Theor. Dyn. Syst.* **1** 361-80
[15] Sattinger D H 1985 *Stud. Appl. Math.* **72** 65-86
[16] Falkensteiner P and Grosse H 1986 *J. Phys. A: Math. Gen.* **19** 2983-9
[17] Giachetti R and Johnson R 1984 *Phys. Lett.* **102A** 81-2
[18] Tu G-z and Men D Z 1989 *Acta Math. Appl. Sin.* **5** 101-8
[19] Jaulent M and Miodek K 1976 *Lett. Math. Phys.* **1** 243-50
[20] Yang C N 1987 *Commun. Math. Phys.* **112** 205-16
[21] Tu G-z 1988 On complete integrability of a large class of finite-dimensional Hamiltonian systems *Preprint*